Revealed Preferences and Utility Functions

Econ 3030 Fall 2025

Lecture 2

Outline

1. Weak Axiom of Revealed Preference
2. Equivalence between Axioms and Rationalizable Choices.
3. An Application: the Law of Compensated Demand.
4. Utility Function

Definitions From Last Class

- A preference relation $\succsim$ is a complete and transitive binary relation on X.
  - describes DM's ranking of all possible pairs of options.
- A choice rule for X is a correspondence $C : 2^X \setminus \{\emptyset\} \to 2^X$ such that $C(A) \subset A$ for all $A \subset X$.
  - describes DM's possible choices from each menu.
- Given $\succsim$, the induced choice rule is $C_{\succsim}(A) = \{x \in A : x \succsim y \text{ for all } y \in A\}$.
- A choice rule C is rationalized by $\succsim$ if it equals the induced choice rule for $\succsim$.
- A choice rule C is rationalizable if there exists a $\succsim$ such that $C = C_{\succsim}$.
- Given a choice rule C, its revealed preference relation $\succsim_C$ is defined by

$$x \succsim_C y \text{ if there exists some } A \text{ such that } x, y \in A \text{ and } x \in C(A).$$

  - If x is chosen when y is available, then x is revealed preferred to y.
- Last class' proposition showed that if C is rationalized by $\succsim$, then $\succsim = \succsim_C$.
- **Next Question**: When is C rationalizable?
  - If C is rationalizable, behavior is consistent with rational decision making because choices could have been driven by some unknown preference relation.

Weak Axiom of Revealed Preference

- The following is the condition necessary for a choice rule to be rationalizable.

Axiom (WARP)

A choice rule for X satisfies the weak axiom of revealed preference

$$\text{If} \quad \begin{array}{l} x, y \in A \cap B, \\ x \in C(A), \text{ and} \\ y \in C(B) \end{array} \quad \Rightarrow \quad x \in C(B).$$

- This is sometimes also known as Houthakker Axiom (see Kreps).
- If x could be chosen from A (when y was also available) and y could be chosen from B (when x was also available) then it must be that x could also be chosen from B.
- In other words, if x was revealed at least as good as y, then y cannot be revealed strictly preferred to x.

Consequences of WARP

WARP:

$$\textit{If } x, y \in A \cap B,\ x \in C(A),\ \textit{and } y \in C(B), \qquad \textit{then} \qquad x \in C(B).$$

Exercise

Verify that WARP is equivalent to the following:

$$\text{If } A \cap C(B) \neq \emptyset, \qquad \text{then} \qquad C(A) \cap B \subseteq C(B).$$

Exercise

Suppose $X = \{a, b, c\}$ and assume $C(\{a, b\}) = \{a\}$, $C(\{b, c\}) = \{b\}$, and $C(\{a, c\}) = \{c\}$. Prove that if C is nonempty, then it must violate WARP. [Hint: Is there any value for $C(\{a, b, c\})$ which will work?]

WARP and Rationalizable Choice Rules

Theorem

Suppose C is nonempty. Then C satisfies WARP if and only if it is rationalizable.

- This gives necessary and sufficient conditions for a choice rule to look as if the decision maker is using a preference relation to generate her choice behavior via the induced choice rule.
  - The preference relation must be a revealed preference by last class' result (if a choice rule is rationalized by some preference, then this preference is a revealed preference).

REMARK

Rationality is equivalent to WARP; thus, one can verify whether or not DM is rational by verifying whether or not her choices obey WARP.

- Proof strategy: One direction is for you and the other for me.

Question 5, Problem Set 1.

Prove that if C is rationalizable, then it satisfies WARP.

WARP Implies Rationalizable I

Step 1: show that if WARP holds then $\succsim_C$ is a preference (remember, $x \succsim_C y$ if $\exists A$ s.t. $x, y \in A$ and $x \in C(A)$).

Proof.

Need to show that $\succsim_C$ is complete and transitive, so it is a preference.

- Let $x, y \in X$. Since C is nonempty, either $x \in C(\{x, y\})$ or $y \in C(\{x, y\})$.
  - Then either $x \succsim_C y$ or $y \succsim_C x$. This proves $\succsim_C$ is complete.
- For transitivity, suppose $x \succsim_C y$ and $y \succsim_C z$. Need to show $x \succsim_C z$.
  - $x \succsim_C y$ means there exist a menu A_{xy} with $x, y \in A_{xy}$ and such that $x \in C(A_{xy})$.
  - $y \succsim_C z$ means there also exists a menu A_{yz} with $y, z \in A_{yz}$ and such that $y \in C(A_{yz})$.
  - Since $C(\{x, y, z\})$ is nonempty, there are three cases:
    - Case 1: $x \in C(\{x, y, z\})$. Then we are done as $x \succsim_C z$.
    - Case 2: $y \in C(\{x, y, z\})$. Observe $x, y \in \{x, y, z\} \cap A_{xy}$, $x \in C(A_{xy})$, and $y \in C(\{x, y, z\})$. By WARP, we must have $x \in C(\{x, y, z\})$ and we are done as $x \succsim_C z$.
    - Case 3: $z \in C(\{x, y, z\})$. Observe $y, z \in A_{yz} \cap \{x, y, z\}$, $y \in C(A_{yz})$ and $z \in C(\{x, y, z\})$. Then, WARP implies $y \in C(\{x, y, z\})$. Now apply Case 2.
  - We have $x \succsim_C z$ in all three cases, thus $\succsim_C$ is transitive. $\square$

WARP Implies Rationalizable II

Step 2: show that if WARP holds then $\succsim_C$ rationalizes C.

Proof.

We need to prove that

$$C(A) = C_{\succsim_C}(A) = \{x \in A : x \succsim_C y \text{ for all } y \in A\}$$

- First, show that $C(A) \subseteq C_{\succsim_C}(A)$:
  - Suppose $x \in C(A)$.
  - Then for any $y \in A$, $x \succsim_C y$, since $x, y \in A$. So $C(A) \subseteq C_{\succsim_C}(A)$
- Now show that $C_{\succsim_C}(A) \subseteq C(A)$.
  - Suppose $x \in C_{\succsim_C}(A)$: for any $y \in A$, there exists some B_{xy} such that $x \in C(B_{xy})$.
  - Since C is nonempty, fix some $z \in C(A)$.
  - WARP applied to $x, z \in B_{xz} \cap A$, $x \in C(B_{xz})$, and $z \in C(A)$ delivers $x \in C(A)$.
  - So $C_{\succsim_C}(A) \subseteq C(A)$.
- Therefore: $C(A) \subseteq C_{\succsim_C}(A)$ and $C_{\succsim_C}(A) \subseteq C(A)$, so we conclude that $C(A) = C_{\succsim_C}(A)$.

$\square$

WARP and Classic Demand Theory

- Classic Demand Theory studies consumers that maximize their utility function subject to their budget constraint.
  - In the next few weeks, we will do this using calculus, but some conclusions can be obtained by observing consumers' choices.

- n goods: consumption $\mathbf{x} \in X = \mathbb{R}^n_+$, prices $\mathbf{p} \in \mathbb{R}^n_{++}$, and income $w \in \mathbb{R}_+$

Definition

A Walrasian demand function maps price-wage pairs to consumption bundles:

$$x^* : \mathbb{R}^n_{++} \times \mathbb{R}_+ \to \mathbb{R}^n_+ \text{ such that } \mathbf{p} \cdot x^*(\mathbf{p}, w) \leq w$$

- Thus $x^*(\mathbf{p}, w) \in B_{\mathbf{p},w} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{p} \cdot \mathbf{x} \leq w \text{ and } x_i \geq 0\}.$
- The definition assumes a unique choice from a given budget set (why?).

Choice Over a Budget Set

Since $B_{\mathbf{p},w}$ represents the menus of affordable consumption bundles:

$$x^*(\mathbf{p}, w) = C(B_{\mathbf{p},w})$$

Properties of Demand Functions

Definition

A Walrasian demand function is homogeneous of degree zero if

$$x^*(\alpha \mathbf{p}, \alpha w) = x^*(\mathbf{p}, w) \text{ for all } \alpha > 0.$$

- In words: nominal price changes have no effect on optimal consumption choices.

Definition

A Walrasian demand function satisfies Full Expenditure if

$$\mathbf{p} \cdot x^*(\mathbf{p}, w) = w.$$

- The consumer spends all of her income (this is sometimes also called Walras' Law for individuals).

Weak Axiom on Budget Sets

Definition

A Walrasian demand function satisfies the weak axiom of revealed preference if, for all pairs $(\mathbf{p}, w)$ and $(\mathbf{p}', w')$:

$$\begin{array}{c} \mathbf{p}' \cdot x^*(\mathbf{p}, w) \leq w' \\ \text{and} \\ x^*(\mathbf{p}, w) \neq x^*(\mathbf{p}', w') \end{array} \quad \Rightarrow \quad \mathbf{p} \cdot x^*(\mathbf{p}', w') > w$$

- Equivalently (recall that here the choice rule is single valued):
  $$C(B_{\mathbf{p},w}) \in B_{\mathbf{p}',w'} \text{ and } C(B_{\mathbf{p}',w'}) \neq C(B_{\mathbf{p},w}) \quad \Rightarrow \quad C(B_{\mathbf{p}',w'}) \notin B_{\mathbf{p},w}$$
- Suppose prices and income change from $(\mathbf{p}, w)$ to $(\mathbf{p}', w')$.
  - The old consumption bundle $x^*(p, w)$ is still affordable since $\mathbf{p}' \cdot x^*(\mathbf{p}, w) \leq w'$...
    - ... yet the consumer changes her choice because $x^*(\mathbf{p}, w) \neq x^*(\mathbf{p}', w')$.
  - Then the new choice could not have been affordable in the old situation,
    - because otherwise she would have chosen $x^*(\mathbf{p}', w')$ while facing $(\mathbf{p}, w)$.

Exercise

Verify that the original WARP axiom, imposed on the limited choice data included in x^*, is equivalent to the stated weak axiom of revealed preference.

Weak Axiom and Compensated Demand

Definition

The pairs $(\mathbf{p}, w)$ and $(\mathbf{p}', w')$ are a compensated price change if

$$\mathbf{p}' \cdot x^*(\mathbf{p}, w) = w'.$$

- A compensated price change gives the consumer enough income so that at the new prices she can still purchase the bundle she chose before.

Question 8, Problem Set 1.

Suppose a Walrasian demand function x^* is homogeneous of degree zero and satisfies Full Expenditure. Prove that if x^* satisfies WARP for all compensated price changes, i.e.

$$\left.\begin{array}{c} \mathbf{p}' \cdot x^*(\mathbf{p}, w) = w' \\ \text{and} \\ x^*(\mathbf{p}, w) \neq x^*(\mathbf{p}', w') \end{array}\right. \quad \Rightarrow \quad \mathbf{p} \cdot x^*(\mathbf{p}', w') > w$$

then it satisfies WARP for all price changes (even the uncompensated ones).

- If the weak axiom of revealed preference holds for these special price changes, then it holds for all price changes.

Law of Compensated Demand

Proposition (Law of Compensated Demand)

Suppose $x^*(\mathbf{p}, w)$ is homogeneous of degree zero and satisfies Full Expenditure. Then the weak axiom of revealed preferences is satisfied if and only if, for any compensated price change $(\mathbf{p}, w)$ and $(\mathbf{p}', w')$ with $w' = \mathbf{p}' \cdot x^*(\mathbf{p}, w)$,

$$(\mathbf{p}' - \mathbf{p}) \cdot [x^*(\mathbf{p}', w') - x^*(\mathbf{p}, w)] \leq 0,$$

with strict inequality if $x^*(\mathbf{p}, w) \neq x^*(\mathbf{p}', w')$.

- Roughly, price and compensated demand move in opposite directions: if price goes up, demand goes down.

Remarks

- This is true *only* for compensated demand.
- We will, in a few lectures, prove a differential version of this result (negative semidefiniteness of the Slutsky matrix) using calculus. The point here is that the law of compensated demand results from homogeneity of degree zero, full expenditure, and the weak axiom of revealed preference. The calculus stuff is extraneous.

WARP implies the Law of Compensated Demand

Proof.

Suppose the weak axiom of revealed preference is satisfied.

The result is immediate if the demands are equal, so wlog assume $x^*(p, w) \neq x^*(p', w')$.

Since $\mathbf{p}' \cdot x^*(\mathbf{p}, w) = w'$ and $w' = \mathbf{p}' \cdot x^*(\mathbf{p}', w')$, by Full Expenditure, we have

$$\mathbf{p}' \cdot [x^*(\mathbf{p}', w') - x^*(\mathbf{p}, w)] = 0$$

Since $\mathbf{p}' \cdot x^*(\mathbf{p}, w) \leq w'$, the weak axiom implies $\mathbf{p} \cdot x^*(\mathbf{p}', w') > w$

Thus: $\mathbf{p} \cdot [x^*(\mathbf{p}', w') - x^*(\mathbf{p}, w)] > 0$

So: $$(\mathbf{p}' - \mathbf{p}) \cdot [x^*(\mathbf{p}', w') - x^*(\mathbf{p}, w)] = \underbrace{\mathbf{p}' \cdot [x^*(\mathbf{p}', w') - x^*(\mathbf{p}, w)]}_{=0} - \underbrace{\mathbf{p} \cdot [x^*(\mathbf{p}', w') - x^*(\mathbf{p}, w)]}_{>0} < 0$$ $\square$

Law of Compensated Demand implies WARP

Proof.

By Question 8 in PS1, it suffices to verify the weak axiom of revealed preference for compensated price changes.

Suppose

$$\mathbf{p}' \cdot x^*(\mathbf{p}, w) = w' \text{ and } x^*(\mathbf{p}, w) \neq x^*(\mathbf{p}', w');$$

We need to show that $\mathbf{p} \cdot x^*(\mathbf{p}', w') > w$.

Then, the law of compensated demand states:

$$\begin{aligned} 0 &> (\mathbf{p}' - \mathbf{p}) \cdot [x^*(\mathbf{p}', w') - x^*(\mathbf{p}, w)] \\ &= \underbrace{\mathbf{p}' \cdot x^*(\mathbf{p}', w')}_{=w'} - \underbrace{\mathbf{p}' \cdot x^*(\mathbf{p}, w)}_{=w'} - \mathbf{p} \cdot x^*(\mathbf{p}', w') + \underbrace{\mathbf{p} \cdot x^*(\mathbf{p}, w)}_{=w} \\ &= w - \mathbf{p} \cdot x^*(\mathbf{p}', w'). \end{aligned}$$

□

Preferences and Utility Functions

- So far, we have discussed the relationship between consumers' preference relations and their behavior.
- Traditionally, economists often model consumers using a utility function u.
  - For example: the consumer chooses, among all affordable bundles, one that maximizes utility:
$$C_u(\text{Budget Set}) = \{\mathbf{x} \in A : u(\mathbf{x}) \geq u(\mathbf{y}) \text{ for all } y \in \text{Budget Set}\}$$
- Next, we look for conditions such that preferences and utility functions are equivalent objects.

Questions

- What does it mean for a function to be consistent with a preference relation?
- When is it possible for a preference relation to be described by such a function?

Preference Representation

Definition

A utility function on X is a function $u : X \to \mathbb{R}$.

Definition

The utility function $u : X \to \mathbb{R}$ represents the binary relation $\succsim$ on X if

$$\mathbf{x} \succsim \mathbf{y} \quad \Leftrightarrow \quad u(\mathbf{x}) \geq u(\mathbf{y}).$$

- Since a utility function maps consumption bundles to real numbers, it represents preferences when the function's values rank any pair of consumption bundles in the same way the preferences would rank them.
- If it exists, a utility function provides a tremendous simplification: instead of comparing two possibly highly dimensional objects using preferences, just look at which of two real numbers is largest.
- Obviously, there are many functions that represent the same preferences:
  - multiply the original function by 3, or add 7 to it, or...
- Under what assumptions can a preference relation be represented by a utility function?

Preferences, Utility Functions, and Ordinality

- Utility functions are unique up to strictly increasing transformations.

Exercise (one direction is not easy)

Suppose $u : X \to \mathbb{R}$ represents $\succsim$. Then $v : X \to \mathbb{R}$ represents $\succsim$ if and only if there exists a strictly increasing function $h : u(X) \to R$ such that $v = h \circ u$.

- This means a utility function represents the preference in an "ordinal" way: the function's value is not important.
  - In other words, only the sign of the difference $u(\mathbf{x}) - u(\mathbf{y})$ matters, not its magnitude.

Existence of a Utility Function and Axioms

- Completeness and transitivity are necessary for existence of a utility function.

Proposition

If there exists a utility function that represents $\succsim$, then $\succsim$ is complete and transitive.

Proof.

Let $u(\cdot)$ be a utility function that represents $\succsim$.

Transitivity.

Suppose $\mathbf{x} \succsim \mathbf{y}$ and $\mathbf{y} \succsim \mathbf{z}$.

Since $u(\cdot)$ represents $\succsim$, we have $u(\mathbf{x}) \geq u(\mathbf{y})$ and $u(\mathbf{y}) \geq u(\mathbf{z})$.Thus, $u(\mathbf{x}) \geq u(\mathbf{z})$.

Since $u(\cdot)$ represents $\succsim$, we conclude $\mathbf{x} \succsim \mathbf{z}$.Hence, $\succsim$ is transitive.

Completeness.

Since $u : X \to \mathbb{R}$, for any $\mathbf{x}, \mathbf{y} \in X$, either $u(\mathbf{x}) \geq u(\mathbf{y})$ or $u(\mathbf{y}) \geq u(\mathbf{x})$.

Since $u(\cdot)$ represents $\succsim$, we have $\mathbf{x} \succsim \mathbf{y}$ or $\mathbf{y} \succsim \mathbf{x}$.Therefore, $\succsim$ is complete. $\square$

Next Class

- Existence of utility functions when X is finite.
- Existence of utility functions when X is infinite: Debreu's Representaion Theorem.